Gravitational lensing evidence computations

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The bending of light by gravitational effects as predicted by general relativity is compared to a Newtonian theory. The mechanism of comparison is Bayesian probability theory. Comparisons are presented with and without the effects of measurement error. The essay begins with a brief overview of Bayes theorem and the associated evidence. Light bending is motivated by dimensional analysis. Finally evidence computations are presented using a late 20th century data set.

Brief background on Bayes theorem

The essence of Bayesian inference is the relationship between posterior, likelihood, prior and evidence. If $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots\theta_m)$ denotes a vector of parameters whose values we would like to infer based upon knowledge of data $\boldsymbol{D}$, then Bayes theorem tell us that

$$p(\boldsymbol{\theta} \mid \boldsymbol{D}) = \frac{1}{Z} \mathcal{L}(\boldsymbol{D} \mid \boldsymbol{\theta}) f(\boldsymbol{\theta})$$

where

$$Z = \int \mathcal{L}(\boldsymbol{D} \mid \boldsymbol{\theta}) f(\boldsymbol{\theta}) d^m \boldsymbol{\theta}$$

is the evidence, $\mathcal{L}(\boldsymbol{D} \mid \boldsymbol{\theta})$ is the likelihood of the data and $f\boldsymbol{\theta})$ is the prior probability distribution. Note that m is the dimension of the parameters. In a Bayesian analysis the likelihood and the prior must be supplied by the user.

If you could remember one thing from conventional probability theory it would be the definition of of conditional probability. This is the same thing as the product rule. The product rule for two propositions A and B is

$$\Pr(A \mid B) = \Pr(A, B) / \Pr(B)$$

where Pr denotes probability, $\Pr(A \mid B)$ reads probability of A given B and $\Pr(A,B)$ reads probability of A and B. We can clearly also write

$$\Pr(B \mid A) = \Pr(B, A) / \Pr(A)$$

But since the probability of A and B is the same thing as the probability of B and A it follows that

$$\Pr(A \mid B) = \frac{\Pr(B \mid A) \Pr(A)}{\Pr(B)}.$$

This is Bayes theorem in a slightly different form. Now make the changes $A \to \boldsymbol{\theta}$ and $B \to \boldsymbol{D}$

$$\Pr(\boldsymbol{\theta} \mid \boldsymbol{D}) = \frac{\Pr(\boldsymbol{D} \mid \boldsymbol{\theta}) \Pr(\boldsymbol{\theta})}{\Pr(\boldsymbol{D})}.$$

In simple terms this is

$$\text{posterior} = \frac{\text{likelihood } x \text{ prior}}{\text{evidence}}.$$

Another way to think about Bayes theorem is

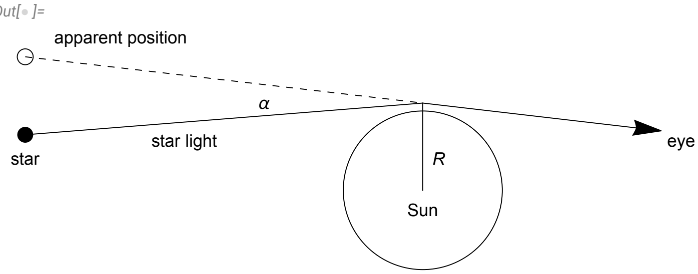
$$\text{likelihood} \times \text{prior} = \text{joint} = \text{evidence} \times \text{posterior}$$
$$\mathcal{L}(\boldsymbol{D} \mid \theta)\, \pi(\boldsymbol{\theta}) = p(\boldsymbol{\theta}, \boldsymbol{D}) = Z p(\boldsymbol{\theta} \mid \boldsymbol{D})$$

You the analyst must supply the likelihood and prior. The Bayesian calculus dictates the evidence and posterior.

Application of dimensional analysis

The gravitational field of the sun bends light which passes near to it. We might expect the amount of bending (the angle α in the accompanying figure) to depend upon the closest point of approach R, the mass of the sun M, the speed of light c and the Newtonian gravitational constant G. We do not include the Planck constant in our list because the theory of general relativity (GR) is not a quantum theory. The dimensional information that we have is summarized in the following table.

Out[ ]=



There are five quantities and three independent dimensions. The Buckingham Pi theorem tells us that the solution can be expressed in terms of two dimensionless groups. One group is the bending angle $\Pi_1=\alpha$ (it is dimensionless). The other group can be found by inspection. It is $\Pi_2 = GM/(c^2 R)$. The formal result of the Pi theorem is

$$\alpha = \phi\left(\frac{GM}{c^2 R}\right)$$

where ϕ is a function that cannot be determined from dimensional analysis. If we make the observation that the gravitational constant G and the solar mass M must occur as a product then our dimensional table will have one less row and we can infer from dimensional analysis that

$$\alpha = \text{consatnt}\ \frac{GM}{c^2 R}.$$

Evidence computations

We consider a problem from the work of Roberto Trotta .

Reference. Roberto Trotta (2017), *Bayesian Methods in Cosmology*, https://arxiv.org/pdf/1701.01467.pdf, exercise 4.6 (ii).

In 1919 two expeditions sailed from Britain to measure the light deflection from stars behind the Sun' s rim during the solar eclipse of May 29. Einstein' s General Relativity predicts a deflection angle

$$\alpha = \frac{4GM}{c^2 R}$$

where G is Newton's constant, c is the speed of light, M is the mass of the gravitational lens and R is the impact parameter. It is well known that this result it exactly twice the value obtained using Newtonian gravity. For $M = M_{sun}$ and R_{sun} one gets from Einstein's theory that $\alpha = 1.74$ arc seconds. The team led by Eddington reported 1.61 ± 0.40 arc seconds (based on the position of 5 stars), while the team headed by Crommelin reported 1.98 ± 0.16 arc seconds (based on 7 stars). What is the Bayes factor between Einstein and Newton gravity from those data? Comment on the strength of evidence.

First check the angle deflection computation using the theory of general relativity theory:

```
c = 299 792 458.0;
M = 1.988435 * 10^30.0;
R = 6957.0 * 10^5.0;
G = 6.67430 * 10^-11;
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$$\alpha Rad = \frac{4 * G * M}{c^2\ R}$$

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8.49012 × 10^-6
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Convert from radians to seconds of angle:

$$\alpha GR = \alpha Rad\ \frac{180}{\pi}\ 60 * 60$$

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αN = 0.5 * αGR
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1.75121
```

```
0.875606
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Evidence computation with no unknown parameters are shown here:

```
αEdd = 1.61; σEdd = 0.40;
αCro = 1.98; σCro = 0.16;
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$$\text{EvidenceGR} = \frac{1}{\sqrt{2\pi}\ \sigma\text{Edd}}\ \text{Exp}\left[-\frac{(\alpha\text{Edd}-\alpha\text{GR})^2}{2\ \sigma\text{Edd}^2}\right]\ \frac{1}{\sqrt{2\pi}\ \sigma\text{Cro}}\ \text{Exp}\left[-\frac{(\alpha\text{Cro}-\alpha\text{GR})^2}{2\ \sigma\text{Cro}^2}\right]$$

$$\text{EvidenceN} = \frac{1}{\sqrt{2\pi}\ \sigma\text{Edd}}\ \text{Exp}\left[-\frac{(\alpha\text{Edd}-\alpha\text{N})^2}{2\ \sigma\text{Edd}^2}\right]\ \frac{1}{\sqrt{2\pi}\ \sigma\text{Cro}}\ \text{Exp}\left[-\frac{(\alpha\text{Cro}-\alpha\text{N})^2}{2\ \sigma\text{Cro}^2}\right]$$

0.840583

2.07929×10^{-11}

The odds in favor of general relativity are :

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OddsGR = EvidenceGR / EvidenceN
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4.04264×10^{10}

These data decisively favor the theory of general relativity.

Now suppose the measurement errors are not reported. Assume a Jeffreys prior on the errors over the range $0.1 < \sigma < 1$. The evidence for general relativity is

$$E_{GR} = \int_{\sigma_{min}}^{\sigma_{max}} \int_{\sigma_{min}}^{\sigma_{max}} \frac{1}{\log\left(\frac{\sigma_{max}}{\sigma_{min}}\right)\sigma_1} \frac{1}{\log\left(\frac{\sigma_{max}}{\sigma_{min}}\right)\sigma_2} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(\alpha_{Edd}-\alpha_{GR})^2}{2\sigma_1^2}\right] \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(\alpha_{Com}-\alpha_{GR})^2}{2\sigma_2^2}\right] d\sigma_1\, d\sigma_2\,.$$

In the above integral the first two factors are the Jeffreys prior on the unknown noise measurement error. The second two factors are the likelihood of the data.

The evidence for the Newtonian theory is

$$E_N = \int_{\sigma_{min}}^{\sigma_{max}} \int_{\sigma_{min}}^{\sigma_{max}} \frac{1}{\log\left(\frac{\sigma_{max}}{\sigma_{min}}\right)\sigma_1} \frac{1}{\log\left(\frac{\sigma_{max}}{\sigma_{min}}\right)\sigma_2} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(\alpha_{Edd}-\alpha_N)^2}{2\sigma_1^2}\right] \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(\alpha_{Com}-\alpha_N)^2}{2\sigma_2^2}\right] d\sigma_1\, d\sigma_2$$

The measurements are independent and uncoupled. Thus the two dimensional integrals can be written as products of one dimensional integrals. This greatly simplifies numeric computation.

The evidence in support of general relativity is:

```
σmin = 0.1; σmax = 1.0;
EvidenceGR =
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$$\frac{1}{2\pi}\ \frac{1}{\text{Log}[\sigma\text{max}/\sigma\text{min}]^2}\ \text{NIntegrate}\left[\frac{1}{\sigma}\ \text{Exp}\left[-\frac{(\alpha\text{Edd}-\alpha\text{GR})^2}{2\ \sigma^2}\right], \{\sigma, \sigma\text{min}, \sigma\text{max}\}\right] *$$

$$\text{NIntegrate}\left[\frac{1}{\sigma}\ \text{Exp}\left[-\frac{(\alpha\text{Cro}-\alpha\text{GR})^2}{2\ \sigma^2}\right], \{\sigma, \sigma\text{min}, \sigma\text{max}\}\right]$$

0.0880326

The evidence in support of the Newtonian theory is:

```
EvidenceN = 1/(2 π) 1/Log[σmax / σmin]^2 NIntegrate[1/σ Exp[-(αEdd - αN)^2/(2 σ^2)], {σ, σmin, σmax}] *
  NIntegrate[1/σ Exp[-(αCro - αN)^2/(2 σ^2)], {σ, σmin, σmax}]
```

$$\text{EvidenceN} = \frac{1}{2\pi}\frac{1}{\text{Log}[\sigma\text{max}/\sigma\text{min}]^2}\,\text{NIntegrate}\left[\frac{1}{\sigma}\,\text{Exp}\left[-\frac{(\alpha\text{Edd}-\alpha\text{N})^2}{2\sigma^2}\right],\{\sigma,\sigma\text{min},\sigma\text{max}\}\right] * \text{NIntegrate}\left[\frac{1}{\sigma}\,\text{Exp}\left[-\frac{(\alpha\text{Cro}-\alpha\text{N})^2}{2\sigma^2}\right],\{\sigma,\sigma\text{min},\sigma\text{max}\}\right]$$

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0.00329603
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The odds ratio in support of general relativity is:

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OddsGR = EvidenceGR / EvidenceN
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26.7086
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This is still quite decisive .

Notes

1) A very readable account of the dimensional analysis aspects of the bending of light by gravity can be found in Sanjoy Mahajan's book T*he Art of Insight in Science and Engineering*.
2)For more on the comparison between light bending as predicted by general relativity and a Newtonian theory see Roberto Trotta's paper "Bayesian Methods in Cosmology".