Sonar Performance Metrics Part 2

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This document illustrates how various types of uncertainty affect the forecasting of sonar performance in naval applications. The first type of uncertainty arises from the fact that we have incomplete knowledge regarding key target kinematic parameters such as range, bearing, depth, heading, speed, etc. In general key sonar performance metrics such as the sonar probability of detection $P(D)$ are dependent upon each of these kinematic parameters. A second type of uncertainty is caused by the actual oceanographic environment in which the sonar operates. At a conceptual level, a sonar makes a mark on a gram or display when the voltage in a detector circuit exceeds a threshold. The probabilities with which these marks occur are determined by the statistics of the noise and signal that the sonar actually experiences. The statistical distribution of the signal and noise fields at the sonar receiver are strongly influenced by a nondeterministic component of ocean sound transmission. Numerous examples are presented.

Background

This document was originally written in 2010 when the author was a scholar in residence at Southeastern Louisiana University. It is presented here in a slightly updated format. In some instances the Mathematica code for computing the figures has been included. The original document contained an introduction and six sections. The section on sonar equations is presented here. The author is currently the Chief Scientist at LogLinear Group, LLC.

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4.0 Sonar equations

4.1 Origin of the equations

In analyzing the performance of the matched filter detector, we obtained an equation which related the signal to noise ratio S/N_0 to a recognition differential at a prescribed level of sonar performance. Specifically we found

$$\frac{S}{N_0} = \frac{(z_{pd} - z_{pfa})^2}{2\,T},$$

where S denotes signal power, N_0 noise spectral density power, both measured at the input to the receiver predetector filter, T is signal integration time and the parameters z_d and z_{pfa} are determined by the required level of sonar performance. A logical question to ask is, "How far can the sonar hear?".

To this end let r denote the distance between the target and the receiver and assume that sound spreads spherically between the target and the receiver. Under these circumstances, the preceding equation can be written in the form

$$\frac{p_s^2}{p_0^2}\frac{r_0^2}{r^2}\frac{p_0^2}{N_0} = \frac{(z_{\text{pd}} - z_{\text{pfa}})^2}{2\,T},$$

where p_s^2 is a measure of the average power radiated by the source, p_0 is a reference pressure and r_0 is a reference distance that is usually assumed to be one meter or one yard. eh reference pressure p_0 is usually assumed to be $10^{-6}\,\mu$ Pa. If we express the foregoing equation on a decibel scale, then we have

$$SL - TL(r) - AN = RD,$$

where

$$SL = 10\log_{10}(p_s^2/p_0^2),\quad TL(r) = -10\log_{10}(r_0{}^2/r^2),$$

$$AN = 10\log_{10}(N_0/p_0^2),\quad RD = 10\log_{10}\left(\frac{(z_{\text{pd}} - z_{\text{pfa}})^2}{2\,T}\right).$$

In the analysis of sonar performance, it is convenient to define a quantity called the signal excess SE via the equation

$$SE = SL - TL(r) - AN - RD.$$

If the signal excess SE is positive, then the signal to noise ratio S/N_0 is in excess of $(z_{\text{pd}} - z_{\text{pfa}})^2/2\,T$ and sonar performance will the required level of performance.

4.2 Empirical lateral range curve for the sonar equation

Let $SE(r)$ denote the expected value of the signal excess (dB) that can be achieved with a sonar against a target at a range r. Experience has shown that the actual (or instantaneous) signal excess $se(r)$ is in many circumstances approximately normally distributed with mean $SE(r)$ and standard deviation σ. A detection is said to occur whenever the actual signal excess $se(r) > 0$. The probability of detection given a target at range r is then

$$P(D \mid r) = P(se(r) > 0) = P\left(\frac{se(r) - SE(r)}{\sigma} > \frac{-SE(r)}{\sigma}\right)$$

$$= \int_{\frac{-SE(r)}{\sigma}}^{\infty} \frac{1}{\sqrt{2\,\pi}}\, e^{-x^2/2}\, dx = \int_{-\infty}^{\frac{SE(r)}{\sigma}} \frac{1}{\sqrt{2\,\pi}}\, e^{-x^2/2}\, dx = \Phi\left(\frac{SE(r)}{\sigma}\right),$$

where $\Phi(y)$ is the cumulative probability distribution function for the standard normal distribution. The false alarm rate associated with this probability of detection is implicitly defined by the recognition differential term is the sonar equation $SE(r)$. The value of the standard deviation σ is often assumed to be about 8 dB. The sensitivity of the lateral range $P(D \mid r)$ to the value of σ is illustrated in figure 4.1. As σ increases from 4 to 12 dB, the range in signal excess over which detection occurs expands. This result follows from the fact that approximately 95% of the area under the standard normal probability density function occurs within two standard deviations of the mean.

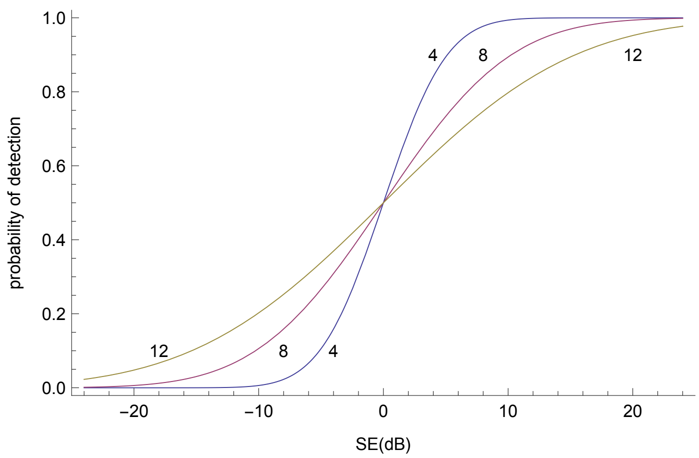


Figure 4.1. Empirical probability of detection for the sonar equation as a function of signal excess *SE* (dB) and three values of the signal excess standard deviation.

4.3 Convolution and the sonar equation

In the empirical model for computing probability of detection from the sonar equation (as described in the previous section), we assumed that the sonar equation was approximately normally distributed. If the statistical distributions of the various terms in the sonar equation are known, then it is possible to directly calculate the actual statistical distribution of the sonar equation using convolution techniques provided that the individual terms in the sonar equation are statistically independent. For instance, the passive sonar equation for an omnidirectional receiver can be written in the form

$$SE(r) = SL - TL(r) - AN - RD,$$

where *SL*, *TL*, *AN* and *RD* respectively denote source level, transmission loss, ambient noise and recognition differential and *r* denotes the range between the target and the receiver. Each of the terms in the sonar equation can be thought of as statistical random variable. If these random variables are independent, then the variance of signal excess σ^2 will be the sum of the variance of the individual terms. The statistical distribution of the signal excess can be found by multiple applications of convolutions of pairs of independent random variables.

To this end suppose that *X* and *Y* are independent random variables with probability density functions $g(x)$ and $h(y)$. We would like to calculate the statistical distribution of the random variable

$$Z = X + Y.$$

The cumulative distribution function $F(z)$ of the random variable *Z* can be found by conditioning with respect to the probability density function $g(x)$:

$$F(z) = P(Z < z) = P(X + Y < z) = \int_{-\infty}^{\infty} P(Y < \mid z - x)\, g(x)\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} h(t)\, dt\, g(x)\, dx.$$

The probability density function $f(z)$ of the random variable *Z* can be found by differentiating the cumulative distribution function $F(z)$:

$$f(z) = \frac{d}{dz} F(z) = \int_{-\infty}^{\infty} h(z-x)\, g(x)\, dx.$$

4.4 Distribution of TL

The acoustic pressure $p(x, y, z, t)$ at any point in the ocean is a solution of the linear wave equation

$$\nabla^2 p = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p,$$

where c denotes the sound velocity at spatial location (x, y, z), t denotes time and ∇^2 is the Laplacian operator in Cartesian coordinates. Solutions to this equation at short wavelengths can be expressed as sums of rays and at long wavelengths as sums of modes (Officer-1958, Brekhovskikh and Lysanov-1982). The complex nature of ocean acoustic propagation often causes the rays (or modes) to have random phase. To this end we will suppose that the pressure at a point is caused by the summation of m rays with equal amplitude α and random phase ϕ_j. If we let r denote the amplitude of the acoustic pressure that results form the summation of these rays, then the probability that r lies in the range r to $r + dr$ is

$$p(r)\, dr = \frac{2}{m\, \alpha^2}\, r\, e^{-r^2/m\, \alpha^2}\, dr,$$

provided that m is large (Skudrzyk-1971). The function $p(r)$ is the probability density function of a Rayleigh random variable with parameter s defined by $s^2 = m\, \alpha^2$. We have encountered the Rayleigh distribution within the context of the envelope detector in section 3.2. In that application, the Rayleigh distribution described the statistical distribution of the noise at the output of an envelope detector in the absence of a signal. The Rayleigh distribution plays an important part in other areas of underwater acoustics as well. The fluctuations in the amplitude of a sound wave reflected from the sea surface can be described by a Rayleigh distribution. The Rayleigh distribution describes the limiting distribution function for the peak values of narrow-band Gaussian noise as the bandwidth approaches zero (Skudrzyk-1971).

The expected value of the amplitude r of the acoustic pressure is the mean of the Rayleigh distribution

$$\mu = E[r] = \int_0^{\infty} r\, p(r)\, dr = \frac{\sqrt{\pi}}{2} \sqrt{m\, \alpha^2}.$$

The expected value of the square of the amplitude r is the second non-central moment

$$m_2 = E\left[r^2\right] = \int_0^{\infty} r^2\, p(r)\, dr = m\, \alpha^2.$$

This result shows that the energies in the rays add and that the resulting total energy is m times the energy in a single ray. The variance of the Rayleigh random variable is

$$\sigma^2 = E\left[(r-\mu)^2\right] = m_2 - \mu^2 = \left(1 - \frac{\pi}{4}\right) m\, \alpha^2.$$

The Rayleigh distribution will result from the addition of components of unequal amplitude provided that the number of each component is large. In this more general case, the parameter of the Rayleigh distribution will be defined by the equation

$$s^2 = \sum_{j=1}^{m} m_j\, \alpha_j^2.$$

The probability density function is

$$p(r) = \frac{2}{s^2}\, r\, e^{-r^2/s^2},$$

and the mean and variance are

$$\mu = \frac{\sqrt{\pi}}{2}\, s, \quad \sigma^2 = \left(1 - \frac{\pi}{4}\right) s^2.$$

The probability that a random amplitude R is less than some specific value of r is the cumulative distribution function

$$F(r) = P(R < r) = 1 - e^{-r^2/s^2}.$$

In many circumstances the quantity of interest is the acoustic intensity $X = R^2$. Acoustic intensity is the square of the amplitude of the acoustic pressure. The probability that a random acoustic intensity X is less than a specific intensity value x is the cumulative distribution function for acoustic intensity defined by

$$G(x) = P(X < x) = P\left(R < \sqrt{x}\right) = \int_0^x \frac{1}{s^2}\, e^{-t/s^2}\, dt.$$

From this it follows that the acoustic intensity is exponentially distributed with probability density function $p(x)$ and mean intensity $s^2 = E[X]$ defined by

$$p(x) = \frac{1}{s^2}\, e^{-x/s^2}.$$

Acoustic transmission loss is simply the acoustic intensity converted to a decibel scale. Specifically if X is the intensity, then the transmission loss Y is

$$Y = -10 \log_{10} X = -\beta \log X,$$

$$\beta = \frac{10}{\log 10} = 4.3429.$$

The cumulative distribution function of the transmission loss is

$$H(y) = P(Y < y) = P\left(X > e^{-y/\beta}\right) = \int_{e^{-y/\beta}}^{\infty} \frac{1}{s^2}\, e^{-x/s^2}\, dx = \int_{-\infty}^{y} p_{TL}(y)\, dy,$$

where $p_{TL}(y)$ is the probability density function

$$p_{TL}(y) = \frac{\exp\left(-\frac{y}{\beta}\right) \exp\left(-\frac{\exp\left(-\frac{y}{\beta}\right)}{s^2}\right)}{\beta\, s^2}.$$

This is the probability density of a Gumbel distribution.The mean and standard deviation of the transmission loss Y are

$$\mu_{TL} = \beta\left[\gamma - \text{Log}\left(s^2\right)\right] = 2.51 - 10\log_{10} s^2,$$

$$\sigma_{TL} = \frac{\pi\,\beta}{\sqrt{6}} = 5.57\text{ dB},$$

where $\gamma = 0577216$ is Euler's gamma constant. A plot of the probability density function $p_{TL}(y)$ of the transmission loss for the case that the mean acoustic intensity $s^2 = 10^{-6}$ is shown in figure 4.2. The mode of the probability density function occurs at $-10\log_{10} s^2 = 60$ dB. The mean is shifted 2.51 dB to the right of the mode. The probability distribution is skewed to the right indicating that there is a significant chance of obtaining large transmission loss values in a multipath environment due to destructive cancellation.

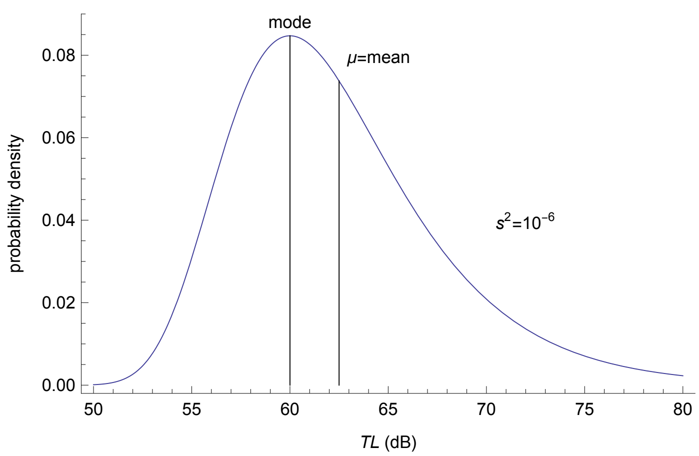


Figure 4.2. Probability density function of acoustic transmission loss level.

If we let N_0 denote the noise intensity spectral level, then the ambient noise spectral level on a decibel scale is $AN = 10\log_{10} N_0$. If we assume that N_0 is exponentially distributed with mean n^2, then the probability density function of the ambient noise level AN is Gumbel distributed with probability density function

$$p_{AN}(y) = \frac{\exp\left(\frac{y}{\beta}\right)\exp\left(-\frac{\exp\left(\frac{y}{\beta}\right)}{n^2}\right)}{\beta\, n^2},$$

where n^2 is the mean noise intensity. In figure 4.3 it has been assumed that $n^2 = 10^8$. The probability density of the ambient noise AN has a long tail to the left. Destructive interference in this case produces lower noise levels. The mean and standard deviation are

$$\mu_{AN} = -\beta\left[\gamma - \text{Log}\left(s^2\right)\right] = -2.51 + 10\log_{10} n^2,$$

$$\sigma_{AN} = \frac{\pi\,\beta}{\sqrt{6}} = 5.57\text{ dB}.$$

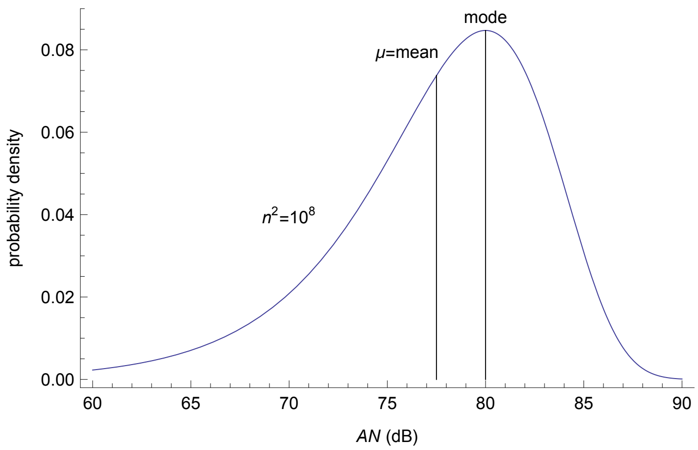


Figure 4.3. Probability density function of ambient noise level.

In order to find the probability of the signal excess *SE* we will first find the probability density of the quantity *TL* + *AN*. This can be accomplished by computing the convolution integral

$$p_{AN+TL}(z) = \int_{-\infty}^{\infty} p_{TL}(z - x)\, p_{AN}(x)\, dx.$$

This integral does not lend itself to closed form evaluation. Figure 4.4 shows a numerical computation of $p_{AN+TL}(z)$ for the case of 60 dB transmission loss and 80 dB ambient noise ($s^2 = 10^{-6}$ and $n^2 = 10^8$). The long tail to the right in the transmission loss has been canceled by the long tail to the left in the ambient noise, resulting in a nearly symmetrical distributing with mean and mode centered on 140 dB. The standard deviation of the distribution of *TL* + *AN* is

$$\sigma_{AN+TL} = \sqrt{\sigma_{AN}^2 + \sigma_{TL}^2} = 7.88 \text{ dB}.$$

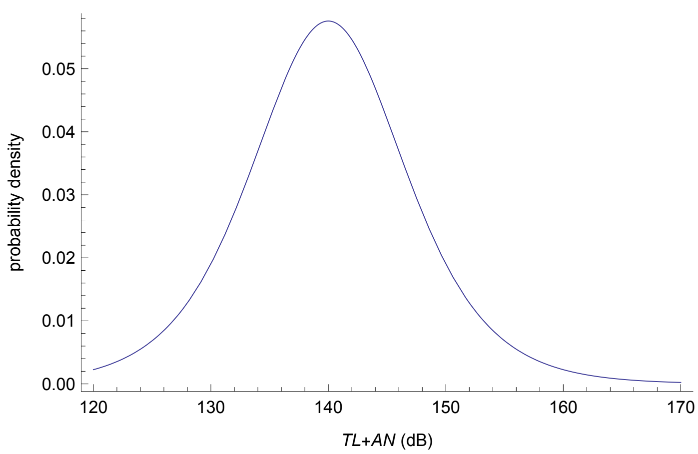


Figure 4.4. Probability density function of transmission loss plus ambient noise level.

In order to provide an example computation of the statistical distribution of the signal excess *SE*, we will assume that the source level *SL* is normally distributed with a mean of 160 dB and a standard deviation of 0.5 dB and that the recognition differential *RD* is a constant 10 dB. The probability density of signal excess can be found by computing the convolution integral

$$p_{SE}(z) = \int_{-\infty}^{\infty} p_{SL-RD}(z - x)\, p_{AN+TL}(x)\, dx,$$

where

$$p_{SL-RD}(x) = \frac{1}{\sqrt{2\pi}\ \sigma_{SL}} e^{-\frac{(x-150)^2}{2\sigma_{SL}^2}},\quad \sigma_{SL} = 1/2.$$

The standard deviation of the signal excess is

$$\sigma_{SE} = \sqrt{\sigma_{AN+TL}^2 + \sigma_{SL}^2} = 7.90 \text{ dB}.$$

The result of this convolution is curve 1 in figure 4.5. Curve 2 in figure 4.5 is a normal distribution centered on a signal excess of 10 dB (*SE*=160-60-80-10) and with a standard deviation of 7.90 dB. The actual distribution of the signal excess is very similar to the normal distribution but less peaked. Both distributions have minimal areas in the tails.

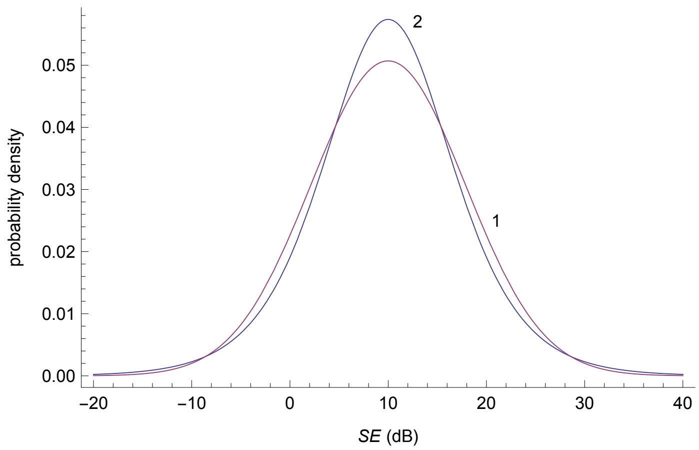


Figure 4.5. Probability density function of signal excess.

In[ ]:=

4.5 Computation of figures 4.1-4.5

Figure 4.1 is computed below:

In[∘]:=
```
Plot[{
  CDF[NormalDistribution[0, 1], se / 4],
  CDF[NormalDistribution[0, 1], se / 8],
  CDF[NormalDistribution[0, 1], se / 12]}, {se, -24, 24}, ⋯]
```

Out[∘]=

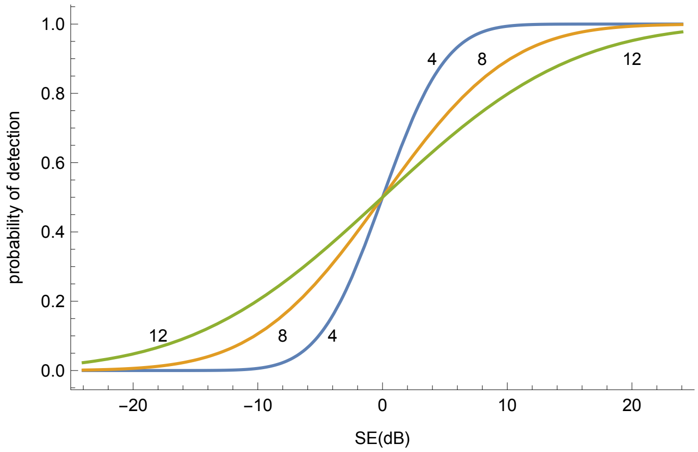


Figure 4.2 is computed below:

In[∘]:=
```
Clear[pTL, y, s2, b];
s2 = 10^-6; b = 10.0 / Log[10];

pTL[y_, s2_] := 1/(s2 b) Exp[(-y/b)] Exp[-1/s2 Exp[-y/b]];
Plot[pTL[y, s2], {y, 50, 80}, ⋯]
```

Out[∘]=

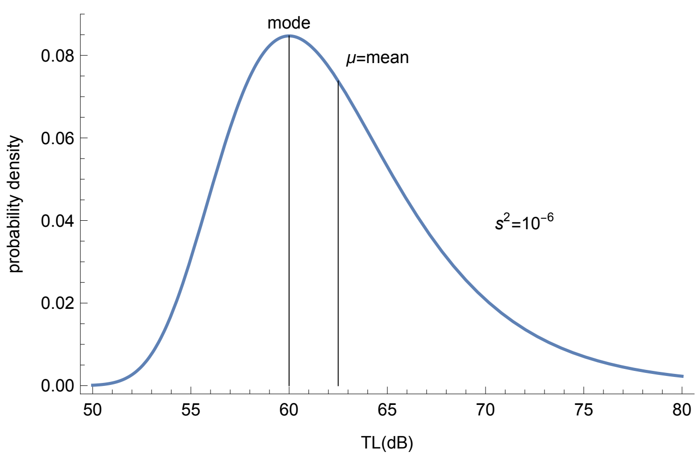


Figure 4.3 is computed below:

In[∘]:=
```
Clear[pAN, y, n2, b];
pAN[y_, n2_] := 1/(n2 b) Exp[(y/b)] Exp[-1/n2 Exp[y/b]];
n2 = 10^8; b = 10.0/Log[10];
```

```
Plot[pAN[y, n2], {y, 60, 90}, ⋯]
```

Out[∘]=

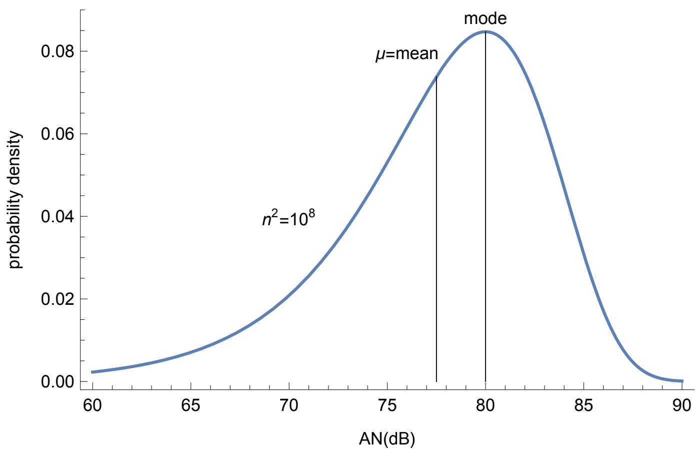


Figure 4.4 is computed below:

In[∘]:=
```
Clear[pTLplusAN];
s2 = 10^-6; n2 = 10^8; b = 10.0/Log[10];
```

```
pTLplusAN[z_] := NIntegrate[pTL[z - y, s2] × pAN[y, n2], {y, 20, 200}];
Plot[pTLplusAN[y], {y, 120, 170}, ⋯]
```

Out[∘]=

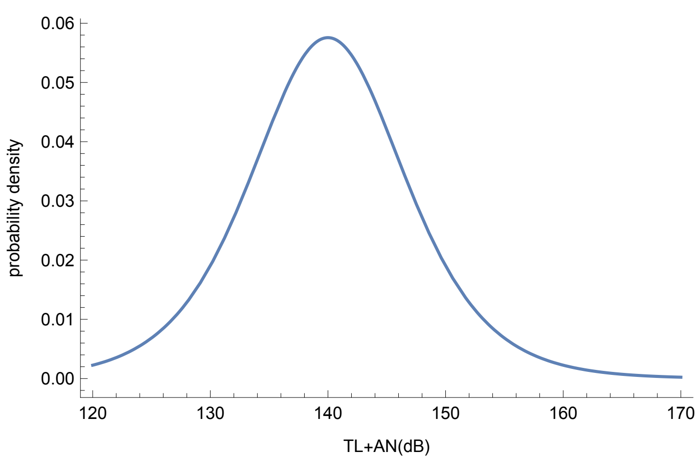
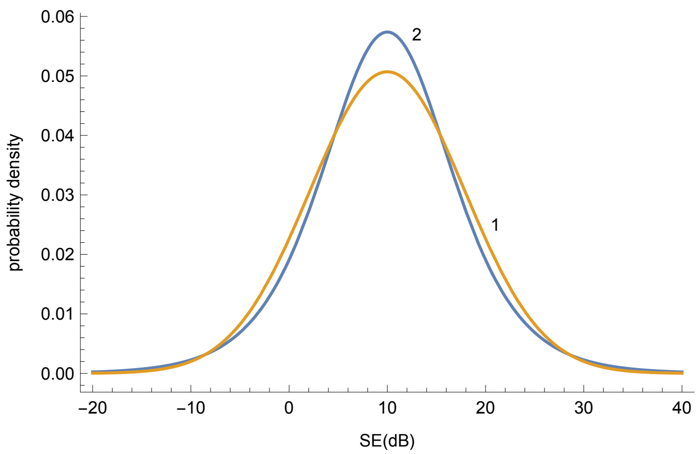


Figure 4.5 is computed below:

In[∘]:=
```
Clear[pTLplusANnumerical, pSE];
RD0 = 10.0; SL0 = 160.0; σSL = 0.5;
pSL[y_, SL0_, RD0_] := 1/(Sqrt[2 π] σSL) Exp[-(y - (SL0 - RD0))^2/(2 σSL^2)];
pTLplusANnumerical =
  Quiet[Interpolation[Table[{y, pTLplusAN[y]}, {y, 80, 200, 1.0}]]];
dy = 0.25;
pSE[z_] :=
  Quiet[dy * Sum[pSL[y + z, SL0, RD0] × pTLplusANnumerical[y], {y, 80, 200, dy}]];
σnorm = 7.87;
Plot[{pSE[y], 1/(Sqrt[2 π] σnorm) Exp[-(y - 10)^2/(2 σnorm^2)]}, {y, -20, 40}, ⋯]
```

Out[∘]=



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