Computation of Sonar Performance Metrics Part 1

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This document illustrates how various types of uncertainty affect the forecasting of sonar performance in naval applications. The first type of uncertainty arises from the fact that we have incomplete knowledge regarding key target kinematic parameters such as range, bearing, depth, heading, speed, etc. In general key sonar performance metrics such as the sonar probability of detection $P(D)$ are dependent upon each of these kinematic parameters. A second type of uncertainty is caused by the actual oceanographic environment in which the sonar operates. At a conceptual level, a sonar makes a mark on a gram or display when the voltage in a detector circuit exceeds a threshold. The probabilities with which these marks occur are determined by the statistics of the noise and signal that the sonar actually experiences. The statistical distribution of the signal and noise fields at the sonar receiver are strongly influenced by a nondeterministic component of ocean sound transmission. Numerous examples are presented.

Background

This document was originally written in 2010 when the author was a scholar in residence at Southeastern Louisiana University. It is presented here in a slightly updated format. In some instances the Mathematica code for computing the figures has been included. The original document contained an introduction and six sections. The introduction and first two sections are presented here. The author is currently the Chief Scientist at LogLinear Group, LLC.

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1.0 Introduction

An important task in naval operational analysis is the forecasting of sonar performance. Questions of prime interest include the following: How far can a particular sonar detect a threat of interest? What is the likelihood that this sonar will detect the threat? Probability theory is used as a basis for answering these questions because of the uncertain nature of the background information upon which the questions are either implicitly or explicitly posed. Uncertainty in the forecasting of sonar performance arises from a variety of inter-related sources. The first type of uncertainty arises from the fact that we have incomplete knowledge regarding key target kinematic parameters such as range, bearing, depth, heading, speed, etc. In general key sonar performance metrics such as the sonar probability of detection $P(D)$ are dependent upon each of these kinematic parameters. When these parameters are only imprecisely known, then we compute a sonar performance metric such as $P(D)$ by marginalization over the joint probability distribution of the kinematic uncertainty.

A second type of uncertainty is caused by the actual oceanographic environment in which the sonar operates. At a conceptual level, a sonar makes a mark on a gram or display when the voltage in a detector circuit exceeds a threshold. The probabilities with which these marks occur are determined by the statistics of the noise and signal that the sonar actually experiences. The statistical distribution of the signal and noise fields at the sonar receiver are strongly influenced by a nondeterministic component of ocean sound transmission. For example, in a multipath environment a passive sonar will often experience a Rayleigh fading signal with a 5.56 dB standard deviation. The sonar designer realizes this and builds a detector which maximizes sonar performance in light of the known statistical distribution of the signal and noise. For the case of the Rayleigh fading signal, the optimum detection algorithm is a matched filter followed by an envelope detector.

Sonar detection thresholds are set based upon the noise. It is not necessary to know exactly how loud the signal is in order to have an optimum detector. However, if we want to forecast sonar performance then we need to predict the signal and noise fields at the sonar receiver. Forecasting sonar performance leads to third type of uncertainty. Due to our imperfect knowledge about the ocean environment (imperfect data bases, unknown locations of noise sources such as merchant ships, etc.), there is a degree of uncertainty associated with our performance prediction. This type of uncertainty deals with a range of possibilities, none of which can be determined from our current state of environmental knowledge.

There are interrelations between each of these types of uncertainty. Imagine a scenario in which the acoustic environment is known with a great degree of certainty but the depth of the threat is unknown. Since the acoustic environment is well known, the transmission loss from the sonar to the range of the tragedy can be predicted with a high degree of certainty. However, since the depth to which this transmission occurs is unknown, the uncertainty in target depth in effect produces a high degree of uncertainty in a forecast of the actual propagation. Target strength if known, will depend strongly upon the relative geometry between the sonar and target. If this geometry is unknown, then the known quantity target strength in effect becomes a statistical quantity due to the underlying geometric uncertainty. This document investigates each of these types of uncertainty and investigates the effects that they have on various measures of sonar performance.

In section 2 we present a brief review of those elements of probability theory that provide the foundation for sonar performance assessment. A measure of sonar performance that is of fundamental importance in this development is the sensor lateral range curve which is the probability of detection for a target located at a known range from a sensor. In mathematical terms it is written $P(D \mid r)$ where D denotes the event detection and r denotes range from the sensor to the target. Using this probabilistic foundation as a basis, we develop a number of different sonar performance metrics and explore the relationships between kinematic uncertainty, sensor lateral range and specific sonar performance metrics. We include a treatment of Bayesian measures of sonar performance. In sections 3 and 4 we address how uncertainty in the oceanographic environment in which the sonar operates effects sonar performance prediction. We use results from signal detection theory (Selin-1965, Whalen-1972) to develop techniques for computing the sensor lateral $P(D \mid r)$ based upon a knowledge of the signal and noise fields at the sonar. Target position is incorporated via the sonar equation (Tucker and Gazey-1977, Urick-1983, Burdick-1984) which is shown to be a natural outgrowth of signal detection theory. In section 5 we show how uncertainty in our knowledge of the environment effects sonar performance prediction. In section 6 we present examples of the computation of sonar performance metrics for several different scenarios of practical interest.

2.0 Measures of sensor performance

2.1 Introduction

In order to obtain estimates of meaningful sonar performance metrics we will draw upon key results from probability theory and the field of Bayesian inference (Jeffreys-1973, Jaynes 2003). A very concise treatment of this topic can be found in (Gregory 2005). Following Jaynes, we employ the notation:

$P(A \mid B)$ = conditional probability for event A, given event B,

$P(AB \mid CD)$ = joint conditional probability for the combined event A and B,

given the combined event C and D.

In terms of the notation commonly used in set theory, $AB = A \cap B$ and $A + B = A \cup B$, where the symbols $\cap$ and $\cup$ respectively denote the operations of intersection and union.

Bayesian inference is concerned with the validity of a set of rival hypotheses $\{H_i\}$, i, 1, 2, .., n, in light of some additional information A and any prior information I. As an example, we may interested in the probability distribution of a target's location in light of the additional knowledge that the target has been detected. The targets probability distribution is in effect a set of hypothesis about where the target is. The additional information is the detection of the target. The prior information could consist of knowledge about the way in which sound propagates through the ocean environment, the reflective or radiation characteristics of the target and performance details about the detecting sonar. In our development and applications, we will assume that the hypotheses H_i are mutually exclusive and exhaustive. The basic rules for manipulating Bayesian probabilities are the sum rule

$P(H_i \mid I) + P(H_i^c \mid I) = 1,$

and the product rule

$P(H_i A \mid I) = P(H_i \mid I) P(A H_i \mid I) = P(A \mid I) P(H_i \mid A I).$

In the most general sense, H_i and A are propositions rather than events. The quantity H_i^c is used to represent the negation of the proposition H_i. It is more general than but analogous to the simple set theory concept of a compliment. Practically speaking, H_i^c can be thought of as the complement of the event H_i.

A set of mutually exclusive hypotheses has the property that $P(H_i H_j \mid I) = 0$ for $i \neq j$. In this case the sum and product rules lead to the generalized sum rule

$P(H_i + H_j \mid I) = P(H_i \mid I) + P(H_j \mid I)$.

Since we have assumed that the hypotheses H_i are mutually exclusive and exhaustive, they in effect span the sample space of interest so that

$$\sum_{i=1}^{n} P(H_i \mid I) = 1.$$

If the background information I does not favor anyone of the hypotheses H_i over another, then

$$P(H_i \mid I) = \frac{1}{n}, \quad i = 1, 2, \ldots n.$$

Conditioning the proposition A on the H_i leads to the following rule for calculating a probability:

$$P(A \mid I) = \sum_{i=1}^{n} P(A \mid H_i I)\, P(H_i \mid I).$$

This last result is also known as marginalization. Bayes theorem follows from the Product rule:

$$P(H_i \mid A I) = \frac{P(H_i \mid I)\, P(A \mid H_i I)}{P(A \mid I)} = \frac{P(H_i \mid I)\, P(A \mid H_i I)}{\sum_{i=1}^{n} P(A \mid H_i I)\, P(H_i \mid I)}.$$

In many circumstances it is not necessary to explicitly retain the dependence of these probabilities upon the prior information I. In fact, the constant appearance of I term is unnecessarily verbose from a notational standpoint. To this end, we state the rules in a more compact form:

Sum rule : $P(H_i) + P(H_i^c) = 1$,

Extended sum rule : $P(H_i + H_j) = P(H_i) + P(H_j)$, for $i \neq j$ and $P(H_i H_j) = 0$,

Product rule : $P(H_i A) = P(H_i)\, P(A \mid H_i) = P(A)\, P(H_i \mid A)$,

Marginalization : $P(A) = \sum_{i=1}^{n} P(A \mid H_i)\, P(H_i)$,

Bayes theorem : $P(H_i \mid A) = \frac{P(A \mid H_i)\, P(H_i)}{P(A)} = \frac{P(A \mid H_i)\, P(H_i)}{\sum_{i=1}^{n} P(A \mid H_i)\, P(H_i)}$.

2.2 Simple examples of Bayes theorem

Bayes theorem is best understood by way of example. Consider the situation of a coastal surveillance craft looking for a small craft illicitly carrying a nuclear weapon. When confronted with a craft carrying the nuclear device, the patrol boat has a sensor that is capable of correctly detecting the presence of the device 99% of the time provided that the device is actually present. The false positive rate for the sensor is 2% and the false negative rate is 1%. The patrol craft searches an area in which 10000 small craft are present. Prior intelligence indicates that one of these craft is carrying an illicit nuclear weapon. In the course of an individual patrol, the patrol craft encounters a small craft and administers the test. A positive result from the test is obtained. A question of obvious importance is the following: Should the patrol craft commander be overly concerned? Bayes theorem provides a mechanism for addressing this question. In order to apply Bayes theorem, we define the hypotheses H_0 and H_1 as follows:

H_0 : The craft is not carrying a nuclear weapon,

H_1 : The craft is carrying a nuclear weapon.

There are two possible decisions that the sensor can make:

D_0 : The craft is not carrying a nuclear weapon,

D_1 : The craft is carrying a nuclear weapon.

These leads to the decision matrix:

Hypothesis / Decision	H_0 is true	H_1 is true
D_0 : Accept H_0	Correct conclusion	Type 2 error : β
D_1 : Accept H_1	Type 1 error : α	Correct conclusion

The performance metrics of the sensor on the patrol craft and the prior information regarding the distribution of the threat imply the following probabilities:

$P(D_1 \mid H_0) = \alpha = 0.02, \;\; P(D_1 \mid H_1) = 1 - \beta = 0.99,$

$P(H_0) = 0.9999, \;\; P(H_1) = 0.0001$

where α is the sensor false positive rate (type 1 error) and β is the sensor false negative rate (type 2 error). The quantity $1 - \beta$ is the power of the sensor.

Bayes theorem implies

$$P(H_1 \mid D_1) = (P(D_1 \mid H_1)\, P(H_1)) / (P(D_1 \mid H_0)\, P(H_0) + P(D_1 \mid H_1)\, P(H_1))$$

$$\frac{0.99\,(0.0001)}{0.02\,(0.9999) + 0.99\,(0.0001)} = 0.004926$$

Even though an alarm has occurred, the probability that the craft is actually carrying a nuclear weapon is quite small. What Bayes theorem tell us in this case is that the combination of a low threat density (1 in 10000 craft) and a relatively high false alarm rate (0.02 per craft encounter) renders the sensor useless as a search tool. The situation drastically changes if the sensor false positive rate can be reduced to 0.01 % . In this case

$$P(H_1 \mid D_1) = \frac{0.99\,(0.0001)}{0.0001\,(0.9999) + 0.99\,(0.0001)} = 0.4975$$

and based upon the occurrence of an alarm, the patrol craft command has about a 50/50 chance of being faced with the craft that is actually carrying the illicit nuclear weapon.

Bayes theorem also applies in situations that are best described by continuous probability density functions. Consider a scenario in which based upon some initial intelligence I, the position of a target at range x with respect to a sensor location is described by the Gaussian probability density function

$$P(x \mid I) = \frac{1}{\sqrt{2\pi}\,\sigma_0} \exp\left[-\frac{(x - x_0)^2}{2\,\sigma_0^2}\right], \;\; -\infty < x < \infty,$$

where $x_0 = 10$ is the mean target position and $\sigma_0 = 4$ is the standard deviation of target position. Both x_0 and σ_0 are assumed to be known as a result of the initial information I. $P(x \mid I)$ is our prior estimate of target position. It is plotted in the accompanying figure. Now suppose that we make a measurement M and that the target is detected at range with positional certainty described by the probability density function

$$P(M \mid x\, I) = \frac{1}{\sqrt{2\pi}\,\sigma_1} \exp\left[-\frac{(x - x_1)^2}{2\,\sigma_1^2}\right], \quad -\infty < x < \infty,$$

where x_1=5 and $\sigma_1 = 2$. The extension of Bayes theorem to continuous probability density functions implies that probability density $P(M \mid x\, I)$ of the target location in light of the measurement M is

$$P(x \mid M\, I) = \frac{P(M \mid x\, I)\, P(x \mid I) \mid}{\int_{-\infty}^{\infty} P(M \mid x\, I)\, P(x \mid I)\, dx}.$$

In explicit terms this is

$$P(x \mid M\, I) = \frac{\exp\left\{-\left[\frac{(x-x_0)^2}{2\,\sigma_0^2} + \frac{(x-x_1)^2}{2\,\sigma_1^2}\right]\right\}}{\int_{-\infty}^{\infty} \exp\left\{-\left[\frac{(x-x_0)^2}{2\,\sigma_0^2} + \frac{(x-x_1)^2}{2\,\sigma_1^2}\right]\right\} dx}.$$

It is often convenient to refer to $P(x \mid M\, I)$ is the posterior target probability density. Figure 2.1 illustrates how the additional information about target location alters the estimate of target position. Initially the target probability density was centered on $x = 10$. The measurement M (data) indicated that the target was centered on $x_1 = 5$ with a reduction in standard deviation from $\sigma_0 = 4$ to $\sigma_1 = 2$. Bayes theorem results in an updated estimate of target location with mean and standard deviation

$$x_2 = \int_{-\infty}^{\infty} x\, P(x \mid M\, I)\, dx = 6.00,$$

$$\sigma_2 = \left[\int_{-\infty}^{\infty} (x - x_2)^2\, P(x \mid M\, I)\, dx\right]^{1/2} = 1.78.$$

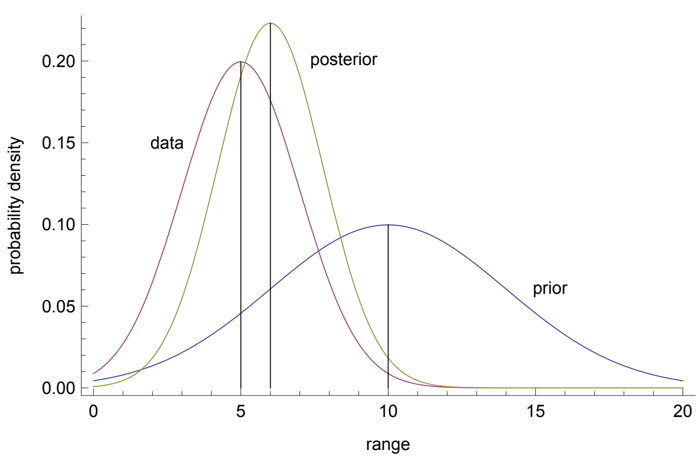


Figure 2.1. Bayesian update of target location.

Now consider a second case in which there is increased uncertainty in the measurement of target position. We will suppose that we make a measurement M and that the target is detected at range with positional certainty described by the probability density function

$$P(M \mid x\, \sigma\, I) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x - x_1)^2}{2\,\sigma^2}\right], \quad -\infty < x < \infty,$$

where σ is described by the uniformly distributed probability density function

$$g(\sigma) = \frac{1}{\sigma_{\max} - \sigma_{\min}}, \quad \sigma_{\min} < \sigma < \sigma_{\max},$$

and zero otherwise. Bayes theorem becomes

$$P(x\,\sigma \mid M\,I) = (P(M \mid x\,\sigma\,I)\,P(x \mid I) \mid g(\sigma)) \Big/ \left(\int_{-\infty}^{\infty} \int_{\sigma_{\min}}^{\sigma_{\max}} P(M \mid x\,\sigma\,I)\,P(x \mid I)\,g(\sigma)\,d\sigma\,dx\right).$$

The posterior target range distribution is obtained by marginalizing $P(x\,\sigma \mid M\,I)$ across σ:

$$P(x \mid M\,I) = \int_{\sigma_{\min}}^{\sigma_{\max}} P(x\,\sigma \mid M\,I)\,d\sigma.$$

Figure 2.2 shows a computation with $\sigma_{\min} = 2$ and $\sigma_{\max} = 5$. Due to the increased uncertainty in the measurement M, the posterior target range distribution is more like the prior distribution than it was in the previous example. The posterior target range distribution is now less peaked and it has a longer tail to the right as evidenced by the shift in the mean (vertical line in figure 2) to the right of the mode. The mean and standard deviation of the posterior range distribution are now respectively 8.25 and 2.59. Before they were 6.0 and 1.78.

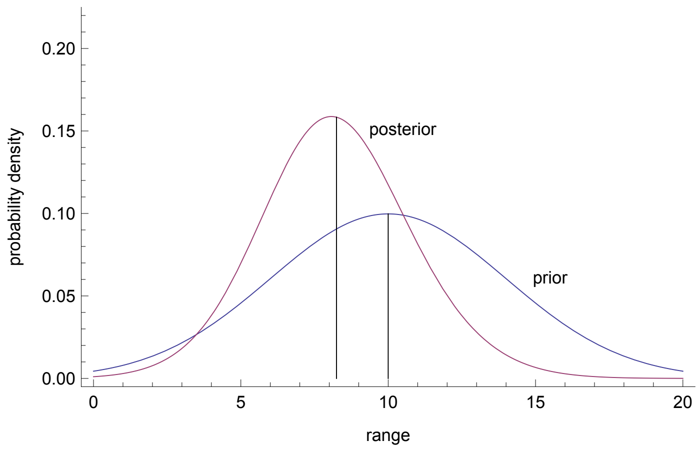


Figure 2.2. Bayesian update of target location with increased uncertainty in the observed data.

2.3 Direct measures of sonar performance

A measure of sonar performance that is of fundamental importance is probability of detection for a target located at a known range from a sensor. This quantity is also referred to as the lateral range curve (Koopman 1980). In mathematical terms it is written $P(D \mid r)$ where D denotes the event detection and r denotes range from the sonar (or sensor) to the target. In situations where sonar performance depends on the bearing of the target relative to the sonar location we write $P(D \mid r\,\theta)$ where θ is the bearing of the target relative to receiver. In general probability of detection versus range can depend on a variety of other factors :

1) Sensor parameters: location, depth, heading relative to north and speed : (x, y, z_0, ψ, v),

2) Target parameters: range from receiver, bearing relative to receiver, depth, heading with respect to north and speed: (r, θ, z, ϕ, u),

3) Other parameters including the time τ which we use to indicate that performance depends on the

specific environmental conditions that are in effect at the time of the prediction.

In addition we always have some prior information about the scenario being addressed. Thus in the most general case we have $P(D \mid x\, y\, r\, \theta\, z\, \phi\, I)$.

In the analysis that follows we will usually assume that sensor parameters are known with complete certainty. If we characterize our uncertainty in target location by the probability density function $P(r, \theta, z)$, then the probability of detection at a given receiver location (x, y) can be computed through the process of marginalization:

$$P(D \mid x\, y) = \int_0^{2\pi} \int_0^{\infty} \int_{z_{min}}^{z_{max}} P(D \mid x\, y\, r\, \theta\, z)\, P(r, \theta, z)\, dz\, dr\, d\theta.$$

If target depth is independent of target position and if target position is characterized by a uniform probability density distribution out to some range r_{max} measured with respect to the receiver location (x, y), then

$$P(D \mid x\, y\, r_{max}) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{r_{max}} \int_{z_{min}}^{z_{max}} P(D \mid x\, y\, r\, \theta\, z)\, P(r)\, f(z)\, dz\, dr\, d\theta,$$

where $f(z)$ is the target depth probability density function and

$$P(r, \theta) = \frac{1}{2\pi} \frac{r}{\frac{1}{2} r_{max}^2} = \frac{1}{2\pi} P(r).$$

If sensor performance is azimuthally independent and if the target is at a known depth, then $f(z) = \delta(z - z_t)$ where $\delta(z)$ is the Dirac delta function and

$$P(D \mid x\, y\, r_{max}) = \frac{1}{\frac{1}{2} r_{max}^2} \int_0^{r_{max}} P(D \mid x\, y\, r\, z_t)\, r\, dr.$$

In terms of a simpler notation this suppresses the sensor position dependence, probability of detection can be written

$$P(D) = \frac{1}{\frac{1}{2} r_{max}^2} \int_0^{r_{max}} P(D \mid r)\, r\, dr.$$

Clearly $P(D)$ is very sensitive to the choice of r_{max}. In fact if r_{max} is large, then $P(D)$ will be small irrespective of the characteristics of the lateral range curve $P(D \mid r)$.

An alternate measure of sonar performance that can be computed from the sensor lateral range $P(D \mid r)$ is the sensor sweep width $W = 2\, W_{1/2}$ where the sensor half sweep width is defined by

$$W_{1/2} = \int_0^{\infty} P(D \mid r)\, dr.$$

The sweep width W is a metric of sonar performance that does not depend on the target probability density $P(r)$. Sweep width plays a prominent part in search theory (Koopman 1980). Consider the problem of detecting a target that is located in a region of area L^2. Suppose that the detection system moves through this region on random course with speed v and with sweep width W. The probability of detecting the target in a small time interval of length Δt is equal to the ratio of the area swept out by the detecting system in time Δt to the area of the search region. This ratio is $\Delta t\, v W / L^2$. The probability of not detecting the target during this time interval is $1 - \Delta t\, v W / L^2$. The probability of not detecting the target in n independent intervals of length Δt is

$$P_{\text{miss}} = \left(1 - \Delta t\, v W / L^2\right)^n.$$

If we define $t = n\,\Delta t$, then the probability of at least one target detection by time t is

$$P_d(t) = 1 - P_{\text{miss}} = 1 - \exp\left[-\frac{t\, v W}{L^2}\right],$$

provided that n is large. The above result is referred to as the formula of random search (Koopman 1980). Its validity depends upon three assumptions. First the target's probability density must be uniformly distributed in the search region. Second, the searcher's path through the search area must be random in the sense that its different segments are placed independently of one another. Third, the searcher will always detects the target within the lateral range $W/2$ on either side of the search path and never at distances beyond this lateral range.

If the search area is rectangular in shape and is searched in an organized fashion with non-overlapping sweeps of width W, then detection can be more rapidly achieved than as predicted by the formula of random search. In this case, the time required to completely sweep the search area is $t_s = L^2 / v W$, and the probability of detection by time t is

$$P_d(t) = \frac{t\, v W}{L^2},$$

and unity at larger times. If we approximate the exponential term in the former result for $P_d(t)$ with a power series to first order in time t, then the latter result for $P_d(t)$ is obtained.

2.4 Bayesian characterizations of performance

In the characterization of sonar performance, we normally think in terms of the sensor lateral range $P(D \mid r)$. Lateral range is the answer to the question, "What is the probability of detection for my sensor given a target at range r?". An alternate measure of sonar performance is the posterior range distribution $P(r \mid D)$. It is the answer to the question, "What is the probability of range given that I make a detection?". The two quantities $P(r \mid D)$ and $P(D \mid r)$ are related by the Bayesian relationship

$$P(r \mid D) = \frac{P(D \mid r)\, P(r)}{P(D)} = \frac{P(D \mid r)\, P(r)}{\int_0^\infty P(D \mid r)\, P(r)\, dr},$$

where $P(r)$ is the prior target range distribution. As previously discussed, $P(D)$ is very sensitive to the choice of the prior range distribution $P(r)$. If the target location probability as described by $P(r)$ is spread out over a wide area then the probability of detection $P(D)$ will be very small. The posterior range distribution $P(r \mid D)$ is not in general very sensitive to the choice of the prior range distribution. The insensitivity of $P(r \mid D)$ to the choice of $P(r)$ follows from the fact that $P(r)$ appears in both the numerator and denominator of the defining expression for $P(r \mid D)$. This is Bayes theory saying, "If you had a detection, then the target had to be close enough for you detect it, regardless of where you thought it was in the first place.".

The prior target range distribution in some circumstances may form the basis for alternate performance measures. One of these is the likely detection range $E[r]$ defined as the mean of the prior target range distribution:

$$\bar{r} = E[r] = \int_0^\infty r\, P(r \mid D)\, dr.$$

Another is the likely detection probability p_d defined as the expected value of the probability of detection $P(D \mid r)$

$$p_d = E[P(D \mid r)] = \int_0^\infty P(D \mid r)\, P(r \mid D)\, dr.$$

2.5 Measures of sensor performance in the presence of multiple targets

Target probability density is not necessarily the appropriate measure of target location with which to characterize sonar detection performance. Imagine two regions with the same spatial and temporal characteristics. Region 1 contains a single target whose location is described by the uniform probability density function $P(x, y)$. Region 2 contains n targets ($n > 1$) where each target operates independently from the others and the location of each of these targets is individually described by the uniform probability density function $P(x, y)$. Clearly a sonar system that operates in region 2 will have a higher probability of detecting a target by virtue of the increased target density in region 2 versus region 1.

Let $P(D \mid r)$ denote the sonar lateral range curve at radial distance r from a point of interest. In probabilistic terms $P(D \mid r)$ is the probability of detection (event D) given the target range r. Suppose that a single target with uniform probability density is located in the circular shaped region $0 < r < r_{\max}$, where $r_{\max}$ is a distance chosen based upon the characteristics of the sonar. Under these circumstances, the target probability density can be written in terms of the range r as,

$$P(r \mid r_{\max}) = \frac{r}{\frac{1}{2} r_{\max}^2},\ 0 < r < r_{\max}.$$

The probability of detecting this single is target is

$$p = P(D \mid r_{\max}) = \int_0^{r_{\max}} P(D \mid r)\, P(r \mid r_{\max})\, dr.$$

Now suppose that there are n independent targets present in the circular shaped region $0 < r < r_{\max}$. The sonar now has n independent chances to make target detections. The probability that the sonar makes at least one detection is

$$p_n = 1 - (1 - p)^n = 1 - q^n,$$

where $q = 1 - p$ is the probability of miss.

Let us suppose that the density (but not probability density) of targets in the region about the sonar is constant and is described by the parameter β measured in number of targets per unit area. Then the expected number of targets in the circular region $0 < r < r_{\max}$ is $\lambda = \pi\, r_{\max}^2\, \beta$. If N denotes the actual number of targets present in the region at a particular instance, then N is a Poisson random variable with mean λ and probability mass function $f(n)$ defined by

$$f(n) = \exp(-\lambda)\,\frac{\lambda^n}{n!},\; n = 0, 1, 2, \ldots$$

The expected value of making at least one detection is $E[p_n]$ can be found by computing

$$E[p_n] = \sum_{n=0}^{\infty} (1 - q^n)\, f(n) = 1 - \exp(-\lambda) \sum_{n=0}^{\infty} \frac{q^n \lambda^n}{n!} = 1 - \exp(-\lambda) \exp(\lambda\, q) = 1 - \exp(-\lambda\, p).$$

In terms of the definition of p and $P(r \mid r_{\max})$ this is

$$E[p_n] = 1 - \exp\left[-\pi\, r_{\max}^2\, \beta \int_0^{r_{\max}} P(D \mid r)\, \frac{r}{\frac{1}{2}\, r_{\max}^2}\, dr\right] = 1 - \exp\left[-2\,\pi\,\beta \int_0^{r_{\max}} P(D \mid r)\, r\, dr\right].$$

The expected value of at least one detection $E[p_n]$ has some very interesting properties. First of all it is a probability in the formal sense that its value lies between zero and one. It does not go to zero as $r_{\max}$ increases thereby as does $P(D \mid r_{\max})$. Furthermore, as a measure of sonar effectiveness, $E[p_n]$ rewards long range sonars and penalizes short ranges sonars since the integral quantity

$$A = 2\,\pi \int_0^{r_{\max}} P(D \mid r)\, r\, dr$$

will be large for a long range sonar and small for a short range sonar. This quantity has the dimensions of area. In fact, we can interpret the quantity A as the area swept out by the sonar in an azimuthally invariant environment. To this end, it is convenient to write

$$E[p_n] = 1 - \exp[-\beta\, A(r_{\max})].$$

The area $A(r_{\max})$ is the two-dimensional analog of the sonar sweep width. Using A as the measure of effectiveness avoids the problem of having
to choose a sonar specific value of $r_{\max}$. In calculating A, you just integrate out in range until $P(D \mid r)$ is vanishingly small.

2.6 Computation of figures 2.1 and 2.2

Figure 2.1

Clear definitions:

```
Clear[p0, p1, s0, s1, x0, xmax, norm, x2];
```

The prior probability density function is:

```
p0[x_] := 1/(Sqrt[2 π] s0) Exp[-(x - x0)^2/(2 * s0^2)]
```

The likelihood of the data is:

```
In[ ]:= p1[x_] := 1/(Sqrt[2 π] s1) Exp[-(x - x1)^2/(2 * s1^2)]
```

The Bayesian evidence (normalization) is:

In[∘]:=
```
s0 = 4.0; x0 = 10.0;
s1 = 2.0; x1 = 5.0; xmax = 30.0;
norm = Integrate[p0[x] × p1[x], {x, 0, xmax}]
```
Out[∘]=
```
0.0477296
```

The posterior probability density function is:

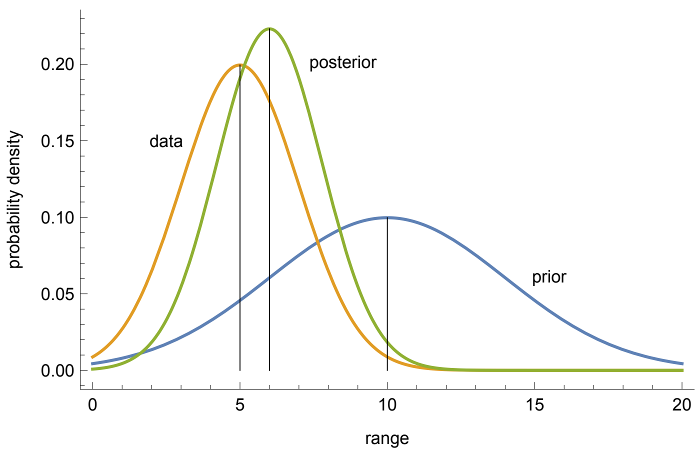
In[∘]:=
$$p2[x\_] := \frac{\left(\frac{1}{\sqrt{2\,\pi}\,s0}\,\mathrm{Exp}\left[-\frac{(x-x0)^2}{2*s0^2}\right]\right)\left(\frac{1}{\sqrt{2\,\pi}\,s1}\,\mathrm{Exp}\left[-\frac{(x-x1)^2}{2*s1^2}\right]\right)}{norm}$$

The posterior mean is given by:

In[∘]:=
```
x2 = NIntegrate[x * p2[x], {x, 0, xmax}]
```
Out[∘]=
```
6.00257
```

Figure 2.1 can be computed via the following:

In[∘]:=
```
Plot[{p0[x], p1[x], p2[x]}, {x, 0, 20}, ⋯ + ]
```
Out[∘]=



In[∘]:=
Figure 2.2

Now we consider a second example (figure 2.2) .

Clear definitions:

In[∘]:=
```
Clear[p0, p1, g, s, s0, s1, smin, smax, x0,
  smin, smax, norm, p2, p2marginal, x2, data, p2marginalA];
```

The prior and the likelihood of the data re respectively:

In[∘]:=
```
p0[x_] := 1/(Sqrt[2 π] s0) Exp[-(x - x0)^2/(2 * s0^2)]
p1[x_, s_] := 1/(Sqrt[2 π] s) Exp[-(x - x1)^2/(2 * s^2)]
g[s_] := 1 / (smax - smin)
```

$$p0[x\_] := \frac{1}{\sqrt{2\pi}\ s0}\ \mathrm{Exp}\left[-\frac{(x - x0)^2}{2 * s0^2}\right]$$

$$p1[x\_, s\_] := \frac{1}{\sqrt{2\pi}\ s}\ \mathrm{Exp}\left[-\frac{(x - x1)^2}{2 * s^2}\right]$$

The evidence (normalization) is:

In[∘]:=
```
s0 = 4.0; x0 = 10.0;
s1 = 2.0; x1 = 7;
smin = 2; smax = 5;
norm = NIntegrate[p0[x] × p1[x, s] × g[s], {x, x0 - 3 s0, x0 + 3 s0}, {s, smin, smax}]
```

Out[∘]=
```
0.0638496
```

The posterior and marginal are:

In[∘]:=

$$p2[x\_, s\_] := \frac{\left(\frac{1}{\sqrt{2\pi}\ s0}\ \mathrm{Exp}\left[-\frac{(x-x0)^2}{2*s0^2}\right]\right)\left(\frac{1}{\sqrt{2\pi}\ s}\ \mathrm{Exp}\left[-\frac{(x-x1)^2}{2*s^2}\right]\right) g[s]}{norm}$$

```
p2marginal[x_] := NIntegrate[p2[x, s], {s, smin, smax}]
```

The mean of the posterior marginal is:

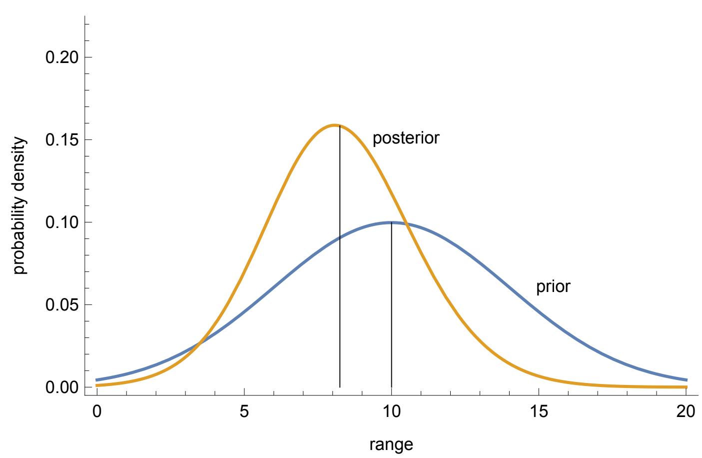
In[∘]:=
```
data = Table[{x, p2marginal[x]}, {x, 0, 20, 0.05}];
p2marginalA = Interpolation[data, InterpolationOrder → 1];
x2 = NIntegrate[x * p2marginalA[x], {x, 0, 20}]
```

Out[∘]=
```
8.24565
```

Figure 2.2 is given by the following:

In[∘]:=
```
Plot[{p0[x], p2marginal[x]}, {x, 0, 20}, ⋯]
```

Out[∘]=



3.0 Signal detection

3.1 Signal known exactly

In order to begin our discussion we will examine the problem of detection from the standpoint of the sonar receiver.The sonar receiver must decide whether a signal of interest is present or not.The sonar receiver observes $x(t)$, the output of a noisy channel for a time interval $(0, T)$. The problem for the receiver is to determine whether or not a signal $s(t)$ is present at the input to the channel. If the signal is present, the channel adds noise $n(t)$ to the signal $s(t)$. If the signal is not present, then the output of the channel is simply the noise $n(t)$. The alternatives that the receiver faces can be written in the form:

$H_0 : x(t) = n(t), \ 0 < t < T,$

$H_1 : x(t) = n(t) + s(t), \ 0 < t < T.$

We will initially assume that the shape of the signal $s(t)$ is exactly known to the sonar receiver. In this case the optimal test statistic $y(T)$ can be shown to be a replica correlator (see Selin 1965 and Whalen 1971) which can be written in the form:

$$y(T) = \int_0^T s(t)\, x(t)\, dt.$$

We will assume that the receiver channel has bandwidth B and that the noise in the receiver channel is white noise that is normally distributed with mean zero and variance $\sigma^2 = N_0 B$ where N_0 is the noise power spectral density which is constant over the band B in accordance with the assumption of white noise. In mathematical terms we can write our description of the noise as

$n(t) = N(0, \sigma^2),$

which reads $n(t)$ is normally distributed with zero mean and variance equal to σ^2.

If we sample the output of the replica correlator at the Nyquist frequency $1/(2B)$ then we will obtain independent samples of identically distributed normal random variables. Either the signal is not present (null hypothesis H_0) or it is present (alternative hypothesis H_1). Thus there are two cases regarding the statistical distribution of the test statistic $y(T)$:

$H_0 : y(T) = n(0, \sigma_y^2),$

$H_1 : y(T) = n(E, \sigma_y^2),$

where E denotes the signal energy and σ_y^2 the variance of the correlator output defined by:

$$E = \int_0^T s(t)^2\, dt, \quad \sigma_y^2 = \frac{E N_0}{2}.$$

A false alarm is said to occur when the statistic $y(T)$ exceeds a threshold K and the signal is not present. A detection is said to occur if $y(T)$ exceeds the threshold K and the signal is actually present. The probabilities of false alarm and detection are then

$$p_{\text{fa}} = \int_K^\infty \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left[-\frac{y^2}{2\sigma_y^2}\right] dy = \int_{z_{\text{pfa}}}^\infty \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{y^2}{2}\right] dy,$$

$$p_d = \int_K^\infty \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left[-\frac{(y-E)^2}{2\sigma_y^2}\right] dy = \int_{z_{\text{pd}}}^\infty \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{y^2}{2}\right] dy$$

where

$$z_{\text{pfa}} = \frac{K}{\sigma_y}, \quad z_{\text{pd}} = \frac{K-E}{\sigma_y}.$$

Elimination of the threshold K from these expressions leads to the relationship

$$\frac{E}{N_0} = \frac{(z_{\text{pd}} - z_{\text{pfa}})^2}{2}.$$

The quantity E/N_0 is a measure of signal to noise ratio (*SNR*). The actual signal to noise ratio at the correlator output is $2\,E/N_0$. The detection threshold DT is defined to be

$$DT = 10 \log_{10}\left[\frac{(z_{\text{pd}} - z_{\text{pfa}})^2}{2}\right] \text{ dB}.$$

Detection threshold is the amount of SNR required to achieve sonar performance at a specified false alarm probability p_{fa} and probability of detection p_d. For instance, a false alarm probability of 10^{-4} requires $z_{\text{pfa}} = 3.72$ and a probability of detection of 0.90 requires z_{pd}=-1.645. This equates to a detection threshold DT of 11.6 dB. For a fixed value of z_{pfa}, i.e. for a fixed false alarm rate, the probability of detection is

$$p_d\left(\frac{E}{N_0}\right) = \int_{z_{\text{pfa}}-z_{\text{snr}}}^\infty \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{y^2}{2}\right] dy = \int_{-\infty}^{z_{\text{snr}}-z_{\text{pfa}}} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{y^2}{2}\right] dy = \Phi(z_{\text{snr}} - z_{\text{pfa}}), \quad .$$

where $\Phi(x)$ is the cumulative normal distribution function and

$$z_{\text{snr}} = \sqrt{\frac{2E}{N_0}}.$$

Since $\Phi(0) = 1/2$, the probability of detection p_d is 0.5 when $z_{\text{snr}} = z_{\text{pfa}}$. For a prescribed probability of detection p_d the condition obtains when

$$\frac{S}{N_0} = \frac{(z_{\text{pd}} - z_{\text{pfa}})^2}{2T}$$

where S defined by

$$S = \frac{1}{T} \int_0^T s(t)^2\, dt$$

is the signal power in the detector band.The recognition differential is defined to be

$$RD = 10 \log_{10}\left[\frac{(z_{\text{pd}} - z_{\text{pfa}})^2}{2T}\right] \text{ dB}.$$

If $z_{\text{snr}} - z_{\text{pfa}} > 2$ then to a good degree of approximation

$$p_d = 1 - \frac{\exp\left[-(z_{\text{snr}} - z_{\text{pfa}})^2/2\right]}{(z_{\text{snr}} - z_{\text{pfa}})\sqrt{2\pi}}.$$

3.2 Noise-like signal

As a second example of signal detection, we consider the detection of a noise-like signal in a noisy background. We will assume that the receiver channel has bandwidth B and that the noise in the receiver channel is white noise that is normally distributed with mean zero and variance $\sigma_n^2 = N_0 B$ where N_0 is the noise power spectral density which is constant over the band B in accordance with the assumption of white noise. The signal in the receiver will also be assumed to be white noise but with variance σ_s^2. The two cases regarding the statistical distribution of the signal in the receiver channel are:

$$H_0 : x(t) = n(0, \sigma_n^2),$$

$$H_1 : x(t) = n(0, \sigma_s^2 + \sigma_n^2),$$

This last equation follows from the fact that when independent Gaussian random variables are added together, the result is a Gaussian random variable whose mean and variance is the sum of the means and variances of the constituents. The optimum test statistic $y(T)$ can be shown to be the energy detector

$$y(T) = \int_0^T x(t)^2 \, dt.$$

In order to evaluate this integral, the output of the receiver channel is sampled at time intervals $\Delta t = 1/(2B)$ for the time interval $(0, T)$ yielding $M = 2BT$ independent samples. Thus in discrete form the test statistic $y(T)$ becomes

$$y(T) = \Delta t \sum_{i=1}^{M} x(t_i)^2$$

Exploiting the fact a sum of squares of independent Gaussian random variables with zero mean are unit variance yields a chi-squared random variable allows us to conclude that

$$H_0 : \frac{1}{\Delta t\, \sigma_n^2} y(T) = \mathrm{X}^2(M),$$

$$H_1 : \frac{1}{\Delta t\left(\sigma_s^2 + \sigma_n^2\right)} y(T) = \mathrm{X}^2(M).$$

The probabilities of detection and false alarm are

$$p_{\text{fa}} = \int_{X_{\text{pfa}}}^{\infty} f(x, M)\, dx, \; p_d = \int_{X_{\text{pd}}}^{\infty} f(x, M)\, dx$$

where $f(x, M)$ denotes the probability density function of a chi-squared random variable with M degrees of freedom:

$$f(x, M) = \frac{x^{M/2-1}\, e^{-x/2}}{\Gamma(M/2)\, 2^{r/2}},\ 0 \le x < \infty, \text{ and}$$

The threshold quantities χ_{pfa} and χ_{pd} are defined by

$$\chi_{\text{pfa}} = \frac{K}{\Delta t\, \sigma_n^2},\ \chi_{\text{pd}} = \frac{K}{\Delta t\, (\sigma_s^2 + \sigma_n^2)}$$

If we eliminate $K/\Delta t$ from the equations defining the quantities χ_{pfa} and χ_{pd}, then we obtain

$$\frac{\sigma_s^2}{\sigma_n^2} = \frac{\chi_{\text{pfa}}}{\chi_{\text{pd}}} - 1.$$

The quantity σ_s^2 / σ_n^2 is a measure of signal to noise ratio (SNR). The detection threshold DT for the energy detector is defined to be

$$DT = 10 \log_{10}\left(\frac{\chi_{\text{pfa}}}{\chi_{\text{pd}}} - 1\right).$$

Detection threshold is the amount of SNR required to achieve sonar performance at a specified false alarm probability p_{fa} and probability of detection p_d. For instance, a false alarm probability of 10^{-4} requires $\chi_{\text{pfa}} = 18.4$ and a probability of detection of 0.90 requires $\chi_{\text{pd}} = 0.21$. This equates to a detection threshold of 19.4 dB. For a fixed value of χ_{pfa}, .e. for a fixed false alarm rate, the probability of detection is

$$p_d\left(\frac{\sigma_s^2}{\sigma_n^2}\right) = \int_{\frac{\chi_{\text{pfa}}}{1+\sigma_s^2/\sigma_n^2}}^{\infty} f(x, M)\, dx.$$

3.3 Signal with unknown phase

In this section we will examine the detection of a sinusoidal signal of known amplitude but unknown phase. We will find that the performance metrics for the detection of this signal are very similar to the case in which the form of the signal is completely known to the receiver. The optimal detector for a signal of known amplitude and frequency but unknown phase in a background of white noise is the envelope detector (Blake 1991). If is

$$x(t) = a \cos(\omega\, t + \phi) + n(t)$$

is the signal plus noise in the receiver channel where $n(t)$ is white noise, then the optimal detection statistic is to compute the signal envelope

$$y(T) = \left[\left[\int_0^T x(t) \cos(\omega\, t)\, dt\right]^2 + \left[\int_0^T x(t) \sin(\omega\, t)\, dt\right]^2\right]^{1/2}.$$

In the absence of signal the test statistic y is Rayleigh distributed with probability density function

$$f(y) = \frac{y}{N} \exp\left(-\frac{y^2}{2\,N}\right),$$

where $N = B\, N_0$ is the noise power in the receiver band, B is the receiver bandwidth and N_0 is the noise spectral density in the receiver band. If the signal is present, then the test statistic y is Rician distributed with probability density function

$$f(y) = \frac{y}{N} \exp\left(-\frac{y^2 + a^2}{2N}\right) I_0\left(\frac{a y}{N}\right),$$

where $I_0(x)$ is the zero order modified Bessel function For a fixed threshold K, the probability of false alarm is

$$p_{\mathrm{fa}} = \int_K^\infty \frac{y}{N} \exp\left(-\frac{y^2}{2N}\right) dy = \exp\left(-\frac{{z_{\mathrm{pfa}}}^2}{2}\right),\; z_{\mathrm{pfa}} = \frac{K}{\sqrt{N}}.$$

The probability of detection is

$$p_d = \int_K^\infty \frac{y}{N} \exp\left(-\frac{y^2 + a^2}{2N}\right) I_0\left(\frac{a y}{N}\right) dy.$$

If we observe that the root mean squared signal power in the receiver band is $S = a^2/2$ denote signal power in the receiver band, then the probability of detection at signal to noise ratio S/N and false alarm rate p_{fa} is

$$p_d\left(\frac{S}{N}\right) = \int_k^\infty \frac{y}{N} \exp\left(-\frac{t^2 + 2\,S/N}{2}\right) I_0\left(\sqrt{2\,S/N}\; t\right) dy,\; k = \sqrt{-2\log(p_{\mathrm{fa}})}.$$

The performance of the three different signal detection models is illustrated in figure x. The false alarm probability is 10^{-6}. Case 1 is detection of a known signal with a matched filter.. Case 2 is detection of a signal of unknown phase with an envelope detector. Case 3 is a detection of a noise-like signal with an energy detector. The time-bandwidth product of the noise-like signal is assumed to be unity. In all cases the background noise is assumed to be white. The performance of the matched filter and the envelope detector are very similar.

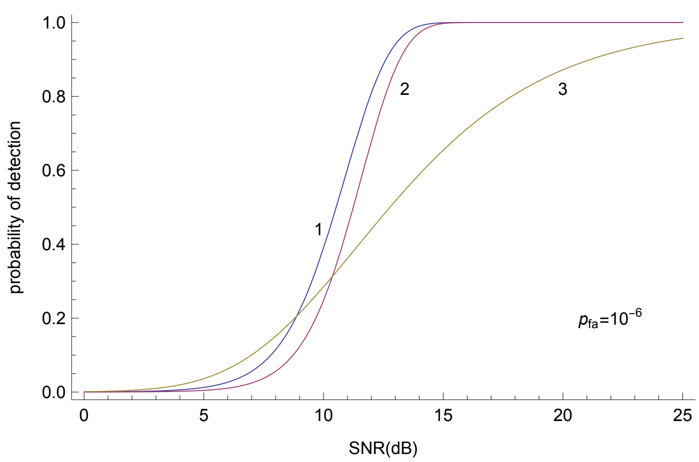


Fig 3.1. Effect of signal to noise ratio on probability of detection for three different signal models. Case 1: signal known exactly. Case2: signal of unknown phase. Case 3: Noise-like signal. In each case an optimal test statistic has been used. Time-bandwidth product for the noise-like signal is unity.

p_d	0.5	0.5	0.9	0.9
p_{fa}	10^{-4}	10^{-6}	10^{-4}	10^{-6}

Known exactly	8.4	10.5	11.0	12.6
Unknown phase	9.4	11.3	11.7	13.2
Noise like	10.9	12.8	19.4	21.1

Table 3.1 Detection thresholds (dB) for three different signal models.

3.4 Odds of sonar success

Odds provide a commonly used, convenient metric for reasoning about stochastic events. If the probability of event E is p, then the odds favoring E are:

$$O = \frac{p}{1-p}.$$

The event we are most interested in is successful performance of the sonar. Consider the two mutually exclusive and exhaustive hypotheses,

H_0: no target is present

H_1: target present,

and two mutually exclusive and exhaustive decisions,

D_0: say no target present

D_1: say target present

then there are four possibilities,

c_{00}:	$D_0 H_0$	correct call, no target
c_{10}:	$D_1 H_0$	false alarm
c_{11}:	$D_1 H_1$	correct call, detection
c_{01}:	$D_0 H_1$	false dismissal.

Following this notational practice, call the probability of event c_{00}, p_{00}.

The event we seek to evaluate is correct sonar operation (S=success), $\{c_{00}, c_{11}\}$ in relation to incorrect sonar operation (F=failure), $\{c_{01}, c_{10}\}$. The odds of sonar success are then

$$O_{ss} = \frac{P(S)}{P(F)} = (P(S \mid H_0)\, P(H_0) + P(S \mid H_1)\, P(H_1)) / (P(F \mid H_1)\, P(H_1) + P(F \mid H_0)\, P(H_0)).$$

If the two hypotheses H_0 and H_1 are equally likely, then $P(H_0) = P(H_1)$, and the odds of sonar success are

$$O_{ss} = \frac{P(S)}{P(F)} = \frac{P(S \mid H_0) + P(S \mid H_1)}{P(F \mid H_1) + P(F \mid H_0)} = \frac{p_{00} + p_{11}}{p_{01} + p_{10}}.$$

Traditionally we specify sonar performance in terms of probability of detection, $p_{11} = p_d$, and probability of false alarm $p_{10} = p_{\text{fa}}$. This means we can re-write the terms

Probability of correct no target: $p_{00} = 1 - p_{\text{fa}}$,

Probability of false dismissal: $p_{01} = 1 - p_d$,

and re-write the odds as,

$$O_{ss} = \frac{1 - p_{\text{fa}} + p_d}{1 + p_{\text{fa}} - p_d}.$$

For the case of the matched filter operating at a prescribed false alarm probability p_{fa}, the odds of sonar success are

$$O_{ss} = \frac{1 - p_{fa} + \Phi\left(\sqrt{\frac{2E}{N_0}} - z_{pfa}\right)}{1 + p_{fa} - \Phi\left(\sqrt{\frac{2E}{N_0}} - z_{pfa}\right)}.$$

where E and N_0 are respectively the signal energy and interference spectral density at the input to the pre-detector filter, $\Phi(y)$ is the cumulative distribution function of a standard Gaussian random variable, and z_{pfa} is the solution to the transcendental equation $p_{fa} = \Phi(-z_{pfa})$.

The odds on a logarithmic scale of the three different signal detection models are illustrated in figure 3.2. The false alarm probability is 10^{-6}. Case 1 is detection of a known signal with a matched filter. Case 2 is detection of a signal of unknown phase with an envelope detector. Case 3 is a detection of a noise-like signal with an energy detector. The time-bandwidth product of the noise-like signal is assumed to be unity. In all cases the background noise is assumed to be white. The performance of the matched filter and the envelope detector are very similar.

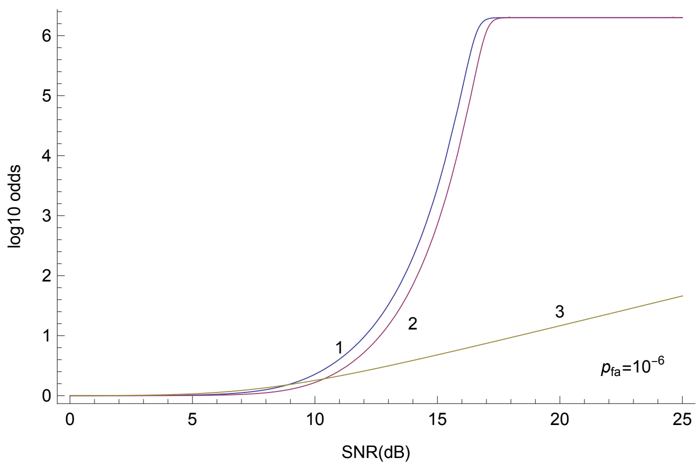


Fig 3.2.Effect of signal to noise ratio on sonar performance odds for three different signal models. The hypotheses H_0 and H_1 have been assumed to be equally likely. Case 1: signal known exactly. Case 2: signal of unknown phase. Case 3: Noise-like signal. In each case an optimal test statistic has been used. Time-bandwidth product for the noise-like signal is unity.

In a search scenario the two hypotheses H_0 and H_1 do not have equal probabilities. It is much more likely that the target is not present, i.e., $P(H_0) >> P(H_1)$. If we define the prior odds ratio to be

$$O_{prior} = P(H_0)/P(H_1),$$

then O_{prior} will be a large positive number, perhaps on the order of 10^4 or more, and the odds of sonar success are

$$O_{ss} = \frac{P(S)}{P(F)} = \frac{P(S \mid H_0)\, O_{prior} + P(S \mid H_1)}{P(F \mid H_1) + P(F \mid H_0)\, O_{prior}} = \frac{(1 - p_{fa})\, O_{prior} + p_d}{(1 - p_d) + p_{fa}\, O_{prior}},$$

$$p_d = \Phi\left(\sqrt{\frac{2E}{N_0}} - z_{\text{pfa}}\right).$$

This last equation shows that if O_{prior} is large, that is if a target is unlikely to be present, then the odds of sonar success O_{ss} are largely determined by p_{fa} and are insensitive to the signal-to-noise ratio E/N_0. A plot of the odds of sonar performance O_{ss} for the matched filter at different values of the prior odds ratio $O_{\text{prior}} = P(H_0)/P(H_1)$ is shown in figure 3.3. When the prior odds ratio is small, the results are very sensitive to SNR.

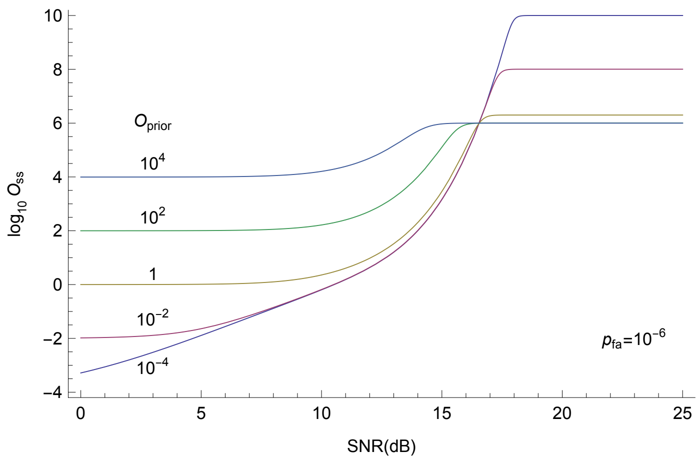


Fig 3.3. Sonar performance odds as a function of SNR for the matched filter at different values of the prior odds ratio.

In[ ]:=

3.5Computations of figures 3.1 - 3.3

In[ ]:=

Figure 3.1

Clear definitions:

In[ ]:=
```
Clear[PdMatchedFilter, PdEnvelopeDetector, PdEnergyDetector];
```

Probability of detection for a matched filter:

In[ ]:=
```
PdMatchedFilter[SNRdB_, Pfa_] := Block[{zpfa, x, snr},
  zpfa = x /. FindRoot[CDF[NormalDistribution[0, 1], x] == 1.0 - Pfa, {x, 4}] ;
  snr = 10^(SNRdB/10);
  1 - CDF[NormalDistribution[0, 1], zpfa - Sqrt[2 snr]]]
```

Probability of detection for an envelope detector:

In[∘]:=
```
PdEnvelopeDetector[SNRdB_, Pfa_] :=
 Module[{a, Threshold},
  a = 10^(SNRdB/20.0);
  Threshold = Sqrt[(-2 Log[Pfa])];
  NIntegrate[
   T * Exp[-(T^2 + 2 a^2)/2] BesselI[0, Sqrt[2] a * T], {T, Threshold, 10 * Threshold}]
 ]
```

$$T * \mathrm{Exp}\left[\frac{-\left(T^2+2\,a^2\right)}{2}\right] \mathrm{BesselI}\left[0,\ \sqrt{2}\ a * T\right]$$

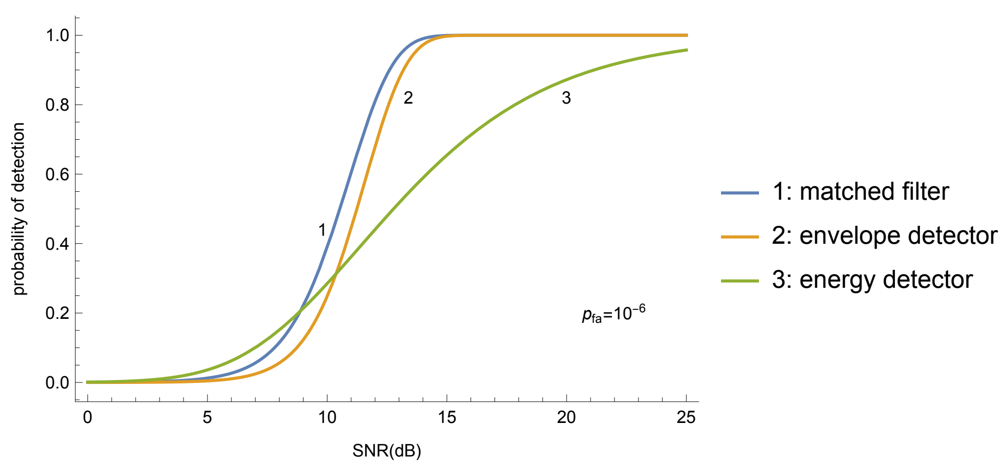
Probability of detection for an energy detector:

In[∘]:=
```
PdEnergyDetector[SNRdB_, Pfa_, MDOF_] := Module[{Xpfa, x, snr},
  Xpfa = x /. FindRoot[CDF[ChiSquareDistribution[MDOF], x] == 1.0 - Pfa, {x, 18}] ;
  snr = 10^(SNRdB/10);
  1 - CDF[ChiSquareDistribution[MDOF], Xpfa/(1 + snr)]]
```

Computation of figure 3.1:

In[∘]:=
```
Pfa = 10^-6; MDOF = 2;
Plot[{PdMatchedFilter[SNRdB, Pfa], PdEnvelopeDetector[SNRdB, Pfa],
  PdEnergyDetector[SNRdB, Pfa, MDOF]}, {SNRdB, 0, 25}, ...]
```

Out[∘]=



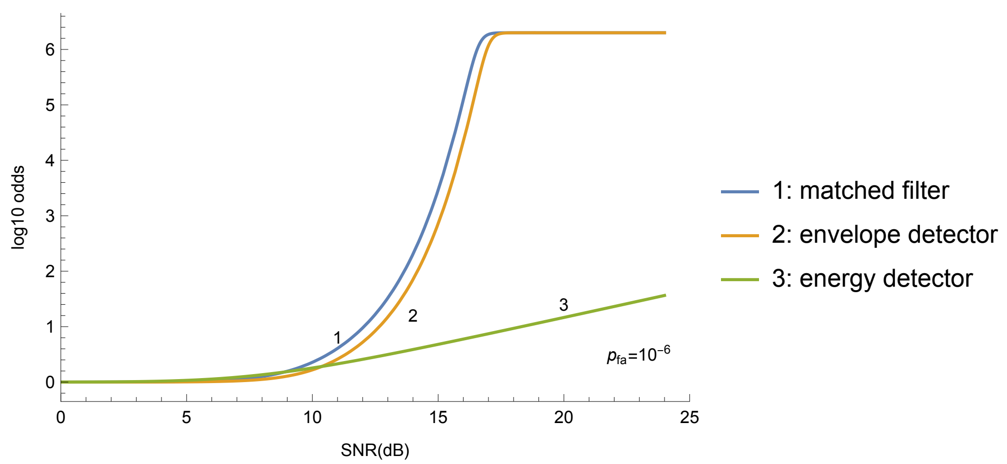
In[∘]:=

Figure 3.2

Computation of figure 3.2:

In[∘]:=
```
Pfa = 10^-6; MDOF = 2;
Plot[{
  Pd = PdMatchedFilter[SNRdB, Pfa];
  Log[10, (1 - Pfa + Pd) / (1 + Pfa - Pd)],
  Pd = PdEnvelopeDetector[SNRdB, Pfa];
  Log[10, (1 - Pfa + Pd) / (1 + Pfa - Pd)],
  Pd = PdEnergyDetector[SNRdB, Pfa, MDOF];
  Log[10, (1 - Pfa + Pd) / (1 + Pfa - Pd)]
 }, {SNRdB, 0, 24}, ⋯ ]
```

Out[∘]=



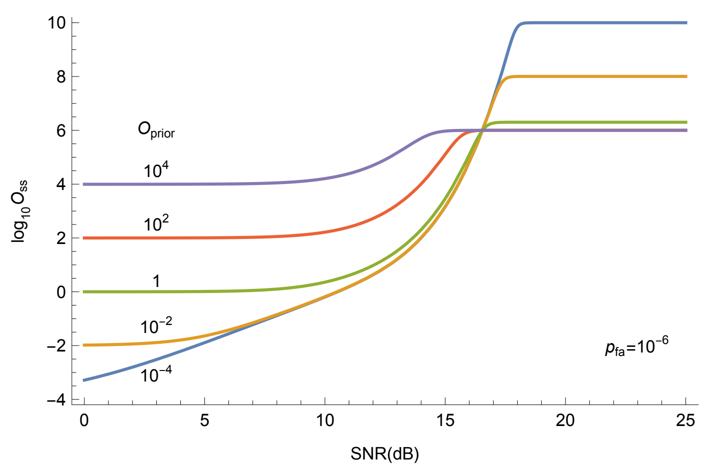
In[∘]:=

Figure 3.3

Computation of figure 3.3:

In[∘]:=
```
Pfa = 10^-6;
Plot[{
   Oprior = 0.0001;
   Pd = PdMatchedFilter[SNRdB, Pfa];
   Log[10, ((1 - Pfa) Oprior + Pd) / ((1 - Pd) + Pfa * Oprior)],
   Oprior = 0.01;
   Pd = PdMatchedFilter[SNRdB, Pfa];
   Log[10, ((1 - Pfa) Oprior + Pd) / ((1 - Pd) + Pfa * Oprior)],
   Oprior = 1;
   Pd = PdMatchedFilter[SNRdB, Pfa];
   Log[10, ((1 - Pfa) Oprior + Pd) / ((1 - Pd) + Pfa * Oprior)],
   Oprior = 100;
   Pd = PdMatchedFilter[SNRdB, Pfa];
   Log[10, ((1 - Pfa) Oprior + Pd) / ((1 - Pd) + Pfa * Oprior)],
   Oprior = 10 000;
   Pd = PdMatchedFilter[SNRdB, Pfa];
   Log[10, ((1 - Pfa) Oprior + Pd) / ((1 - Pd) + Pfa * Oprior)]
  },
  {SNRdB, 0, 25}, ...]
```

Out[∘]=



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